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# Additive deformations of the $r$-matrix algebras

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**Abstract.** We show how to construct new representations of the various $R$-matrix algebras starting from known representations. For linear $r$-matrix algebras we investigate a dynamical $r$-matrix which depends on the spectral parameter and half of the dynamical variables (particle coordinates) only. The Toda lattices and the Henon–Heiles systems illustrate the scheme.

## 1. Introduction

The progress in understanding the algebraic roots of quantum and classical integrability achieved in recent decades has already resulted in the introduction of several new algebraic objects in the framework of the quantum inverse scattering method (QISM), such as the Yang–Baxter equation (YBE) [4, 19], the fundamental commutator relation (FCR) [9], and the reflection equation (RE) [16, 8]. One of the main problems of the QISM to find new representations of $R$-matrix-algebras for a given matrix $R(u)$, since they correspond to new integrable systems.

In the present paper we develop a scheme allowing the construction of new representations of the various $R$-matrix algebras, starting from the known representations. The paper is organized as follows. In section 2 quadratic $R$-matrix algebras and their deformations are described. Examples of such algebras are given in section 3, with applications to the theory of the finite-dimensional integrable system. In section 4 the special deformations of the linear $r$-matrix algebras in two-dimensional auxiliary spaces are discussed. Examples of the integrable systems which are connected with these linear algebras are given in section 5. In the conclusion we discuss some other possibilities of deforming the $R$-matrix algebras, and their applications.

## 2. Deformation of the quadratic $R$-matrix algebras

The standard notations for the basic quadratic $R$-matrix algebra $g_R$ are given via the fundamental commutator relation (FCR) [4, 9]

$$R(u-v)\overset{1}{T}(u)\overset{2}{T}(v)=\overset{2}{T}(v)\overset{1}{T}(u)R(u-v) \tag{2.1}$$

where $\overset{1}{T}=T(u)\otimes I$, $\overset{2}{T}=I\otimes T_0(v)$, and the spectral paramters $u$ and $v$ are associated with the first and the second auxiliary spaces respectively. The operator $T(u)$ is an $N\times N$ matrix in the auxiliary space, with entries $T_{ij}$ being operators in the quantum space. The matrix $R(u)$ is a solution of the YBE, and it acts on the tensor product of two auxiliary spaces. It is easy to see that the matrix trace, $t(u)$, of $T(u)$

$$t(u) = \operatorname{tr} T(u) = \sum_{k=1} T_{kk}(u) \tag{2.2}$$

forms a commutative family of operators

$$[t(u), t(v)] = 0 \tag{2.3}$$

which we will consider as integrals of motion of some quantum integrable system.

Let $T_1(u)$ and $T_2(u)$ be two representations of the algebra (2.1) in the quantum spaces $V_1$ and $V_2$ respectively. Then the matrix $T(u) = T_1(u)T_2(u)$ also gives a representation of the algebra (2.1) in the quantum space $V_1 \otimes V_2$, called the tensor product of the representations $T_1$ and $T_2$ [4,9]. The possibility of multiplying the representations of the algebra (2.1) immediately provides a way of constructing an arbitrary number of new representations from the know representations.

Let the operator $T_0(u)$ be a representation of (2.1). Then one can introduce a modified operator $T(u)$ of the form

$$T(u) = T_0(u) - F(u) \qquad \text{or} \qquad T_0(u) = T(u) \cdot S_1(u) = S_2(u) \cdot T(u). \tag{2.4}$$

Here $F(u)$ is an undefined, but additive, deformation of the matrix $T_0(u)$, and the matrices $S_1(u)$ and $S_2(u)$ are

$$S_1(u) = I + (T_0(u) - F(u))^{-1} \cdot F(u) \qquad S_2(u) = I + F(u) \cdot (T_0(u) - F(u))^{-1}. \tag{2.5}$$

After substitution of equality (2.4) into the equation (2.1) one obtains that the modified operator $T(u)$ satisfies the generalized reflection equation (GRE)

$$\begin{aligned} R(\overset{1}{T} + \overset{1}{F})(\overset{2}{T} + \overset{2}{F}) &= (\overset{2}{T} + \overset{2}{F})(\overset{1}{T} + \overset{1}{F})R \\ &= R(u-v)\,\overset{1}{T}(u)\,\overset{1}{S}_1(u)\,\overset{2}{S}_2(v)\,\overset{2}{T}(v) = \overset{2}{T}(v)\,\overset{2}{S}_1(v)\,\overset{1}{T}(u)R(u-v) \\ &= R(u-v)\,\overset{1}{T}(u)S_{12}(u,v)\,\overset{2}{T}(v) = \overset{2}{T}(v)S_{21}(u,v)\,\overset{1}{T}(u)R(u-v). \end{aligned} \tag{2.6}$$

Here, the matrices $S_{12}$ and $S_{21}$ are

$$\begin{aligned} S_{12}(u,v) &= S_1(u) \otimes S_2(v) \\ &= I + (\overset{1}{T}_0 - \overset{1}{F})^{-1}\,\overset{1}{F} + \overset{2}{F}(\overset{2}{T}_0 - \overset{2}{F})^{-1} + (\overset{1}{T}_0 - \overset{1}{F})^{-1}\,\overset{1}{F}\,\overset{2}{F}(\overset{2}{T}_0 - \overset{2}{F})^{-1} \\ S_{21}(u,v) &= P \cdot S_{12}(v,u) \cdot P \\ &= I + (\overset{2}{T}_0 - \overset{2}{F})^{-1}\,\overset{2}{F} + \overset{1}{F}(\overset{1}{T}_0 - \overset{1}{F})^{-1} + (\overset{2}{T}_0 - \overset{2}{F})^{-1}\,\overset{2}{F}\,\overset{1}{F}(\overset{1}{T}_0 - \overset{1}{F})^{-1} \end{aligned} \tag{2.7}$$

where $P$ is the operator of permutation of the two auxiliary spaces $P(A \otimes B) = (B \otimes A)P$ [4].

The matrices $S_{12}$ and $S_{21}$ are functions of the initial operator $T_0(u)$ and the additive deformation $F(u)$, and obviously depend on dynamical variables. The additive deformation

$F(u)$ was chosen in such a way that the inverse matrix $(T_0 - F)^{-1}$ exists. We can also impose some additional constraints on it. For instance, we can demand that the initial operator $T_0(u)$ and the modified operator $T(u)$ (2.4) obey the FCR (2.1). This condition gives the following equation for the additive deformation $F(u)$

$$R(\overset{1}{F}\overset{2}{T}_0 + \overset{1}{T}_0\overset{2}{F} - \overset{1}{F}\overset{2}{F}) = (\overset{2}{F}\overset{1}{T}_0 + \overset{2}{T}_0\overset{1}{F} - \overset{2}{F}\overset{3}{F})R. \tag{2.8}$$

As the second constraints we can take the condition that the modified matrix $T(u)$ obeys the reflection equation (RE) [16, 8]. Then we have to demand the equality of the matrices $S_{12}$ and $S_{21}$

$$S \equiv S_{12} = S_{21}\,. \tag{2.9}$$

It is an equation for the additive deformation $F(u)$, and hence the modified operator $T(u)$ obeys the reflection equation (RE) in standard form

$$R\,\overset{1}{T}(u)S\,\overset{2}{T}(u) = \overset{2}{T}(v)S\,\overset{1}{T}(u)R. \tag{2.10}$$

The integrable systems corresponding to the RE (2.10) can be defined by (2.2) [16, 8].

As well as for the FCR (2.1) we can consider a similar construction of the additive deformation for the initial algebra defined by the generalized reflection equation (GRE). Let the operator $T_0$ obey the GRE

$$A\,\overset{1}{T}_0(u)B\,\overset{2}{T}_0(v) = \overset{2}{T}_0(v)C\,\overset{1}{T}_0(u)D \tag{2.11}$$

and a modified operator $T(u)$ is introduced by the rule (2.4). After substitution of equality (2.4) into equation (2.11) one obtains that the new operator $T(u)$ satisfies the following GRE

$$A\,\overset{1}{T}(u)S_B\,\overset{2}{T}(v) = \overset{2}{T}(v)S_C\,\overset{1}{T}(u)D \tag{2.12}$$

with the matrices $S_B$ and $S_C$ depending on dynamical variables

$$\begin{aligned} S_B(u,v) &= \overset{1}{S}_1(u)\cdot B(u,v)\cdot \overset{2}{S}_2(v) \\ &= B + (\overset{1}{T}_0 - \overset{1}{F})^{-1}\overset{1}{F}B + B\overset{2}{F}(\overset{2}{T}_0 - \overset{2}{F})^{-1} + (\overset{1}{T}_0 - \overset{1}{F})^{-1}\overset{1}{F}B\overset{2}{F}(\overset{2}{T}_0 - \overset{2}{F})^{-1} \end{aligned} \tag{2.13}$$

$$\begin{aligned} S_C &= \overset{2}{S}_1(v)\cdot C(u,v)\cdot \overset{2}{S}_1(u) \\ &= C + (\overset{2}{T}_0 - \overset{2}{F})^{-1}\overset{2}{F}C + C\overset{1}{F}(\overset{1}{T}_0 - \overset{1}{F})^{-1} + (\overset{2}{T}_0 - \overset{2}{F})^{-1}\overset{2}{F}C\overset{1}{F}(\overset{1}{T}_0 - \overset{1}{F})^{-1}. \end{aligned}$$

Similarly to the quadratic algebra defined by the FCR (2.1) we can introduce some natural restrictions on the additive deformation $F(u)$.

For simplicity, in what follows we restrict ourselves to the two-dimensional auxiliary space and $R$-matrixs of the $XXX$ and $XXZ$ types only

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix} \qquad u \in \mathbb{C} \tag{2.14}$$

where the functions $a(u)$, $b(u)$ and $c(u)$ read as

$$a(u) = 1 + \frac{\eta}{u} \qquad b(u) = 1 \qquad c(u) = \frac{\eta}{u} \qquad \eta \in \mathbb{C}$$
$$a(u) = \sinh(u + \eta) \qquad b(u) = \sinh u \qquad c(u) = \sinh \eta$$

for the $XXX$ and $XXZ$ $R$-matrices, respectively. We will use the standard notations for the entries of $T_0$

$$T_0(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}(u). \tag{2.15}$$

The local integrals of motion $H^{(k)}$ are obtained as the coefficients of the polynomial $t(u)$ (2.2) [4, 9]

$$\begin{aligned} t(u) &= \sum u^k \cdot H^{(k)} \qquad &&\text{for the } XXX \text{ case} \\ t(u) &= \sum \exp(ku) \cdot H^{(k)} \qquad &&\text{for the } XXZ \text{ case.} \end{aligned} \tag{2.16}$$

The determination of the additive deformation $F(u)$ from the conditions (2.8) and (2.9) by the given matrix $T_0(u)$ is a complicated problem which is as difficult as the search for the new representations of the algebras corresponding to the FCR (2.1) or the RE (2.10). Due to this we will construct some special solutions $F(u)$ of the equations (2.9) and (2.8).

Let the matrix $T_0(u)$ obey either the FCR (2.1) or the RE (2.10) with the $R$-matrix (2.14). When the matrix $T_0(u)$ satisfies the RE (2.10), we require also its unitarity

$$T_0^{-1}(-u) \sim T_0(u + \eta).$$

Further simplification arises from the quantum determinant, which is a Casimir operator for the algebras connected with the FCR and the RE [4, 9]. For the FCR it is defined as

$$\begin{aligned} \Delta_0(u) &\equiv \det{}_q T_0(u) \\ &= D(u - \tfrac{1}{2}\eta)A(u + \tfrac{1}{2}\eta) - B(u - \tfrac{1}{2}\eta)C(u + \tfrac{1}{2}\eta) \\ &= A(u - \tfrac{1}{2}\eta)D(u + \tfrac{1}{2}\eta) - C(u - \tfrac{1}{2}\eta)B(u + \tfrac{1}{2}\eta) \\ &= A(u + \tfrac{1}{2}\eta)D(u - \tfrac{1}{2}\eta) - B(u + \tfrac{1}{2}\eta)C(u - \tfrac{1}{2}\eta) \\ &= D(u + \tfrac{1}{2}\eta)A(u - \tfrac{1}{2}\eta) - C(u + \tfrac{1}{2}\eta)B(u - \tfrac{1}{2}\eta). \end{aligned} \tag{2.17}$$

To fix the additive deformation $F(u)$ we will use the following idea. We will choose additive deformations that only slightly deform the quantum determinant. For example, we can try the simplest deformations of the type

$$F(u) = \begin{pmatrix} 0 & 0 \\ f(u)B^{-1} & 0 \end{pmatrix} \tag{2.18}$$

where $f(u)$ is a function of the spectral paramter $u$ only. As the monodromy matrix is

$$T(u) = T_1(u) \cdot T_2(u) \cdots T_n(u)$$

where the $T_k(u)$ obey the FCR (2.1) and can be deformed by the rule (2.18), the deformation (2.18) changes the integrals of motion constructed by (2.16). The quantum determinant of the matrix $T_0(u)$ modified by the rule (2.4) with the additive deformation $F(u)$ (2.18) now reads

$$\begin{aligned}\Delta(u) &= \Delta_0(u) - f(u+\tfrac{1}{2}\eta)B(u-\tfrac{1}{2}\eta)B^{-1}(u+\tfrac{1}{2}\eta)\\ &= \Delta_0(u) - f(u-\tfrac{1}{2}\eta)B^{-1}(u-\tfrac{1}{2}\eta)B(u+\tfrac{1}{2}\eta)\\ &= \Delta_0(u) - f(u-\tfrac{1}{2}\eta)B(u+\tfrac{1}{2}\eta)B^{-1}(u-\tfrac{1}{2}\eta)\\ &= \Delta_0(u) - f(u+\tfrac{1}{2}\eta)B^{-1}(u+\tfrac{1}{2}\eta)B(u-\tfrac{1}{2}\eta)\end{aligned} \tag{2.19}$$

where $\Delta_0(u)$ stands for the quantum determinant of the initial matrix $T_0(u)$. Because the quantum determinant $\Delta(u)$ has to be Casimir operator for the new algebra connected with the FCR or the RE, this equation gives a very strong restriction on the functions $f(u)$ and the entry $(T_0(u))_{12} \equiv B(u)$.

We will also use a more complicated deformation $F(u)$

$$F(u) = \begin{pmatrix} f_+ - f_- & 0 \\ [f + f_+(A+D)f_-(A-D)]B^{-1} & f_+ + f_- \end{pmatrix}(u) \tag{2.20}$$

where we will demand that functions $B(u)$, $f(u)$ and the combinations $(f_\pm(u)[A(u)\pm D(u)])$ do not depend on the spectral parameter $u$. One cannot use the additive deformations (2.18) and (2.20) for an $R$-matrix of the $XYZ$ type, because then $[B(u), B(v)] \neq 0$. A similar restriction holds for the linear $r$-matrix algebras also.

Note that in the theory of quantum groups, where the FCR and the RE do not depend on the spectral parameter $u$, the condition (2.19) is simplest.

## 3. Examples of quadratic $R$-matrix algebras

Here we consider a few integrable systems originated by various additive deformations $F(u)$.

(1) A *singular oscillator* is connected with the $T$-matrix

$$\begin{aligned}T(u) = T_0 + F &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}(u) + \begin{pmatrix} 0 & 0 \\ \mu B^{-1} & 0 \end{pmatrix}\\ &= \begin{pmatrix} u + \{pq\} & q^2 \\ -p^2 & u - \{pq\} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \dfrac{\mu}{q^2} & 0 \end{pmatrix}\end{aligned} \tag{3.1}$$

where braces $\{,\}$ stand for anticommutators and $p, q$ are the canonically conjugate momentum and coordinate of the particle.

The matrices $\sigma_{1,2}T(u)$ and $\sigma_{1,2}T_0(u)$ (3.1) give rise to the Hamiltonians

$$H_0 = p^2 \pm q^2$$
$$H = H_0 + \frac{\mu}{q^2}$$

where $\sigma_i$ are the Pauli matrices.

The operators $T_0(u)$ and $T(u)$ (3.1) obey the FCR with the $R$-matrix of the $XXX$ type (2.14)

(2) *A special case of Neumann's system* is defined on the Lie algebra $e(3)$ [12]. Let the variables $M_\alpha$, $p_\alpha$, $\alpha = 1, 2, 3$ be generators of the Lie algebra $e(3)$ obeying the commutator relations:

$$[M_\alpha, M_\beta] = -\mathrm{i}\varepsilon_{\alpha\beta\gamma} M_\gamma \qquad [M_\alpha, p_\beta] = -\mathrm{i}\varepsilon_{\alpha\beta\gamma} p_\gamma$$
$$[p_\alpha, p_\beta] = 0 \qquad \alpha, \beta = 1, 2, 3.$$

with the special values of the Casimir operators

$$a^2 - p_\alpha p_\alpha = 1 \qquad l = M_\alpha p_\alpha = 0.$$

The initial operator $T_0(u)$ and the modified operator $T(u)$ are defined by

$$T_0(u) = \begin{pmatrix} u^2 + 2M_3 u - M_1^2 - M_2^2 - \frac{1}{4} & bp_+ u + \frac{1}{2}\{p_3, M_+\} \\ bp_- u + \frac{1}{2}\{p_3, M_-\} & b^2 p_3^2 \end{pmatrix} \qquad b \in \mathbb{R} \tag{3.2}$$

and

$$T(u) = T_0(u) + \begin{pmatrix} \dfrac{\mu^2 - \frac{1}{4}}{p_3^2} & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} \qquad \mu \in \mathbb{R} \tag{3.3}$$

where we use the natural notations $M_\pm = M_1 \pm \mathrm{i}M_2$, $p_\pm = p_1 \pm \leq p_2$, and the braces $\{\,,\,\}$ stand for the anticommutator.

The Hamiltonians corresponding to $T_0(u)$ and $T(u)$ (2.16) read as

$$H_0 = M_1^2 + M_2^2 + b^2 p_3^2$$
$$H = H_0 + \frac{\mu^2 - \frac{1}{4}}{p_3^2}.$$

The initial operator $T_0(u)$ (3.2) and the modified operator $T(u)$ correspond to the special case of Neumann's system and obey the FCR (2.1) with the $R$-matrix of the $XXX$ type (2.14). The matrix $T(u)$ was studied in [12].

In what follows we will consider the lattice integrable systems connected with the RE (2.10). The respective monodromy matrices will be constructed by the rule

$$T(u) = K_+(u) \left( \prod_{k=2}^{N-1} L_k(u) \right) K_-(u) \left( \prod_{k=2}^{N-1} L_k(-u) \right)^{-1}. \tag{3.4}$$

Here the matrices $L_k(u)$ and $K_\pm$ obey the FCR (2.1) and RE (2.10) with one $R$-matrix. There are some isomorphisms among the matrices $K_-(u)$ and $K_+(u)$ [16], for instance

$$K_+(u) \equiv K_-^t(-u - \eta) \qquad \text{or} \qquad K_+(u) \equiv \left(K_-^{-1}(-u - \eta)\right)^t \tag{3.5}$$

where $t$ stands for matrix transposition. Because of this we will write the matrix $K_-(u)$ only. The integrals of motion $H^{(k)}$ are obtained by the rule (2.16), as well as for the FCR algebra.

(3) The *Toda lattices* associated with the Lie algebras of $B_n$, $C_n$ and $D_n$ series [2, 6]. We introduce the initial operator $K_A(u, p, q)$

$$K_A(u) = \begin{pmatrix} (u-p)\exp(q) & \exp(2q) \\ u^2 - p^2 & \exp(q)(u+p) \end{pmatrix} \tag{3.6}$$

where $p, q$ are the canonically conjugate momentum and coordinate of the particle. The operator $K_A(u)$ obeys the RE (2.10), where $R = R(u-v)$ and $S = R(u+v)$ with the $R$-matrix of the $XXX$ type (2.14).

According to the rule (2.20) we will consider a modified operator $K_{BC}(u, p, q)$

$$\begin{aligned} K_{BC}(u) &= K_A(u) - F(u) \\ &= K_A + \begin{pmatrix} \dfrac{\alpha}{u} + \beta & 0 \\ \exp(-2q)[\gamma + 2\alpha\exp(q) + 2\beta\exp(q)] & \dfrac{\alpha}{u} - \beta \end{pmatrix}. \end{aligned} \tag{3.7}$$

The operators $K_A$ and $K_{BC}$ can be factorized by three simple factors [16].

Because the quantum determinant of the initial operator $K_A(u, p, q)$ is equal to zero ($\Delta_0(u) = 0$) and the initial operator $K_A(u, p, q)$ obeys unitarity $-A(-u-\eta)$, we can also use a more complicated deformation $F_D(u)$

$$F_D(u) = -\begin{pmatrix} B^{-1}(u)D(u) & B^{-1}(u) + f(u) \\ 0 & A(u)B^{-1}(u) \end{pmatrix}. \tag{3.8}$$

Having this deformation, we obtain a modified operator $K_D(u, p, q)$ [11]

$$K_D(u) = K_A - F_D = \begin{pmatrix} (u-p)e^q + e^{-q}(u+p) & e^{2q} + e^{-2q} - 2 \\ u^2 - p^2 & (u-p)e^{-q} + e^q(u+p) \end{pmatrix}. \tag{3.9}$$

The operators $K_A$, $K_{BC}$ anmd $K_D$ correspond to the Toda lattices associated with the Lie algebras of $A_n$, $B_n$ ($\beta = \gamma = 0$), $C_n$ ($\alpha = \beta = 0$) and $D_n$ series respectively [11].

The operator $K_D$ can be further generalized by the rule (2.20)

$$K_{gD}(u) = K_D - \begin{pmatrix} \dfrac{\alpha}{u} + \beta & 0 \\ \dfrac{1}{\sinh^2 q}[\gamma + 2\alpha\cosh q + 2\beta p \sinh q] & \dfrac{\alpha}{u} - \beta \end{pmatrix}. \tag{3.10}$$

The modified operators $K_{BC}(u)$, $K_D(u)$ and $K_{gD}(u)$ obey the RE (2.10), where $R = R(u-v)$ and $S = R(u+v)$ with the standard $R$-matrix of the $XXX$ type (2.14) [16, 11].

The Hamiltonians for these systems follow by the rule (2.16) from the matrix $T(u)$ (3.4) with the matrices

$$L_k(u) = \begin{pmatrix} u - p_k & -\exp(q_k) \\ \exp(-q_k) & 0 \end{pmatrix}.$$

The matrices $K_\pm(u)$ are constructed from the matrices $K_A$, $K_{BC}$, $K_D$, $K_{gD}$ or the unit matrix.

Among the Hamiltonians there are

$$H_A = \sum_{j=1}^{N} \tfrac{1}{2} p_j^2 + \sum_{j=1}^{N-1} \exp(q_{j+1} - q_j)$$

$$H_{BC} = H_A + \gamma \exp(-2q_1) + [2\alpha + 2\beta p_1] \exp(-q_1)$$

$$H_D = H_A + \exp(-q_1 - q_2)$$

$$H_{gD} = H_D + \frac{\gamma + 2\alpha \cosh q_1 + 2\beta p_1 \sinh q_1}{\sinh^2 q_1}$$

the complete set of Hamiltonians and $K$ matrix, except the matrix $K_A$, was considered in [11].

(4) The *relativistic Toda lattices* associated with the Lie algebras of $B_n$, $C_n$ and $D_n$ series [13]. We start with the initial operator $\widehat{K}_A(u, p, q)$

$$\widehat{K}_A(u) = \begin{pmatrix} \sinh(u-p)\exp(q) & \exp(2q) \\ \sinh(u-p)\sinh(u+p) & \exp(q)\sinh(u+p) \end{pmatrix} \tag{3.11}$$

where $p, q$ are the canonically conjugate momentum and coordinate of the particle. The operator $\widehat{K}(u)$ obeys the RE (2.10), where $R = R(u - v)$ and $S = R(u + v)$ with the $R$-matrix of the $XXZ$ type (2.14).

We can consider an operator $\widehat{K}_{BC}(u, p, q)$ modified by the rule (2.20)

$$\widehat{K}_{BC}(u) = \widehat{K}_A(u) - F(u) \tag{3.12}$$

$$= \widehat{K}_A + \begin{pmatrix} \dfrac{\alpha}{\sinh u} + \dfrac{\beta}{\cosh u} & 0 \\ \exp(-2q)[\gamma + 2\alpha\cosh p \exp(q) + 2\beta \sinh p \exp(q)] & \dfrac{\alpha}{\sinh u} - \dfrac{\beta}{\cosh u} \end{pmatrix}.$$

The operators $\widehat{K}_A$ and $\widehat{K}_{BC}$ can be decomposed into three simple factors [13].

Because the quantum determinant of the initial operator $\widehat{K}(u, p, q)$ is equal to zero ($\Delta_0(u) = 0$) and the initial operator is the unitary $D(u) = -A(-u - \eta)$, we can use a more complicate deformation $F_D(u)$ (3.8), as well as in the non-relativistic case. This deformation gives a modified operator $\widehat{K}(u, p, q)$

$$\widehat{K}_D(u) = \widehat{K}_A(u) + F_D(u)$$

$$= \begin{pmatrix} \sinh(u-p)e^{q} + e^{-q}\sinh(u+p) & e^{2q} + e^{-2q} - \sinh^2 u - 2 \\ \sinh^2 u - \sinh^2 p & \sinh(u-p)\,e^{-q} + e^{q}\sinh(u-p) \end{pmatrix}. \tag{3.13}$$

The operators $\widehat{K}_A$, $\widehat{K}_{BC}$ and $\widehat{K}_D$ correspond to the relativistic Toda lattices associated with the Lie algebras of $A_B$, $B_n$ ($\beta = \gamma = 0$), $C_n$ ($\alpha = \beta = 0$) and $D_n$ series, respectively [13]. The operator $\widehat{K}_D$ can not be generalized by the rule (2.20), as it was in the non-relativistic case, since now the entry $B(u)$ of the matrix $\widehat{K}_D$ depends on the spectral parameter $u$.

The initial operator $\widehat{K}(u)$ and modified operators $\widehat{K}_{BC}(u)$ obey the RE (2.10), where $R = R(u - v)$ and $S = R(u + v)$ with the standard $R$-matrix of $XXZ$ type (2.14) [13].

The Hamiltonians for these systems are constructed by the rule (2.16) from the matrix $T(u)$ (3.4) with the matrices $L(u)$

$$L_k(u) = \begin{pmatrix} \sinh(u - p_k) & -\exp(q_k) \\ \exp(-q_k) & 0 \end{pmatrix}.$$

The matrices $K_\pm(u)$ are constructed from the matrices $\widehat{K}_A$, $\widehat{K}_{BC}$, $\widehat{K}_D$ or the unit matrix.

Some Hamiltonians produced by the scheme we have developed read as

$$\widehat{H}_A = \sum_{j=1}^{N} \exp(p_j)[1 + \exp(q_{j+1} - q_j)]$$
$$\widehat{H}_{BC} = \widehat{H}_A + \gamma \exp(-2q_1) + [2\alpha \cosh p_1 + 2\beta \sinh p_1] \exp(-q_1)$$
$$\widehat{H}_D = \widehat{H}_A + 2\exp(q_1 + q_2) \cosh \tfrac{1}{2}(p_1 + p_2) + \exp(2q_2).$$

the complete set of Hamiltonians is considered in [13].

(5) The *Heisenberg $XXX$ and $XXZ$ model.* Let the L-operators $L_k(u)$ in (3.4) be

$$L_k(u) = \begin{pmatrix} uS_0^{(k)} - S_3^{(k)} & S_-^{(k)} \\ S_+^{(k)} & uS_0^{(k)} + S_3^{(k)} \end{pmatrix}$$
$$\widehat{L}_k(u) = \begin{pmatrix} \sinh u\, S_0^{(k)} - \cosh u\, S_e(k) & S_-^{(k)} \\ S_+^{(k)} & \sinh u\, S_0^{(k)} + \cosh u\, S_3^{(k)} \end{pmatrix} \tag{3.14}$$

where $k$ is the number of the particle in the chain and $S_\alpha^{(k)}$ are operators representing the algebras with the quadratic relations described in [15]. In particular, the operators $S_\alpha$ can be realized by the spin operators $s_\alpha$ [15], for instance

$$S_0 = 1 \qquad S_3 = \eta s_3 \qquad S_\pm = \eta s_\pm$$
$$S_0 = \cosh \tfrac{1}{2}\eta \qquad S_3 = \sinh \tfrac{1}{2}\eta\, s_3 \qquad S_\pm = \sinh \tfrac{1}{2}\eta \cosh \tfrac{1}{2}\eta\, S_\pm \tag{3.15}$$

for the $XXX$ and $XXZ$ chain. This choice corresponds to the ordinary $XXX$ and $XXZ$ spin-$\frac{1}{2}$ chains [5]. The operators $L_k$ and $\widehat{L}_k$ obey the FCR (2.1) with the $R$-matrix of the $XXX$ and $XXZ$ types, respectively.

We will consider the following initial operators for the $XXX$ model

$$K_{A1}(u) = \begin{pmatrix} (uS_0 - S_3)S_- & u^2 S_0^2 - S_3^2 \\ S^2 & S_-(uS_0 + S_3) \end{pmatrix}$$
$$K_{A2}(u) = \begin{pmatrix} (uS_0 - S_3)S_+ & S_+^2 \\ u^2 S_0^2 - S_3^2 & S_+(uS_0 + S_3) \end{pmatrix} \tag{3.16}$$

and for the $XXZ$ model

$$\widehat{K}_{A1}(u) = \begin{pmatrix} (\sinh u\, S_0 + \cosh u\, S_3)S_+ & S_+^2 \\ S_-^2 & S_-(\sinh u\, S_0 - \cosh u\, S_3) \end{pmatrix}$$
$$\widehat{K}_{A2}(u) = \begin{pmatrix} (\sinh u\, S_0 + \cosh u\, S_3)S_+ & S_+^2 \\ \sinh^2 u\, S_0^2 - \cosh^2 u\, S_3^2 & S_+(\sinh u\, S_0 - \cosh u\, S_3) \end{pmatrix} \tag{3.17}$$

which obey the RE (2.10) with the corresponding $R$-matrices.

The modified operators $K_{BC}$ and $\widehat{K}_{BC}$ are constructed from the operators $L_k$ (3.14) and $K_A$ (3.16) and (3.17) by the rule

$$\begin{aligned} K_{BC} &= \alpha K_{A1} + \beta K_{A2} + \gamma[2\sinh u\, S_0 \widehat{L}(u) - \Delta(u) I] \\ \widehat{K}_{BC} &= \alpha \widehat{K}_{A1} + \beta \widehat{K}_{A2} + \gamma[2\sinh u\, S_0 \widehat{L}(u) - \widehat{\Delta}(u) I] \\ &\alpha, \beta, \gamma \in \mathbb{R} \qquad \Delta(u) \equiv \det_q L(u) \qquad \widehat{\Delta}(u) \equiv \det_q \widehat{L}(u). \end{aligned} \tag{3.18}$$

Here $I$ is the unit matrix, $\Delta(u)$ and $\widehat{\Delta}(u)$ are the quantum determinants of the operators $L(u)$ and $\widehat{L}(u)$ respectively.

The operators $K_{A1}$, $K_{A2}$ and $\widehat{K}_{A1}$, $\widehat{K}_{A2}$, $\widehat{K}_{BC}$ can be decomposed into three simple factors [16, 3]. As well as the Toda systems we can also construct the more complicated operators $K_D$ and $\widehat{K}_D$, which are not factorized by simple factors.

Let the deformation read as

$$F_D(u) = -\begin{pmatrix} B^{-1}(u)D(u) & B^{-1}(u) + f(u) \\ 0 & A(u)B^{-1}(u) \end{pmatrix}$$

where the operators $(S_\pm)^{-1}$ are replaced by the operatorx $S_\mp$, respectively. Then the modified operators $K_D$ and $\widehat{K}_D$ are

$$\begin{aligned} K_D(u) &= \begin{pmatrix} uS_0S_1 - \mathrm{i}S_3S_2 & u^2S_0^2 - S_2^2 \\ u^2S_0^2 - S_3^2 & uS_0S_1 + \mathrm{i}S_3S^2 \end{pmatrix} \\ \widehat{K}_D(u) &= \begin{pmatrix} \sinh u\, S_0S_1 + \mathrm{i}\cosh u\, S_3S_2 & \sinh^2 u\,(S_0 - \Delta(u)) - S_2^2 \\ \sinh^2 u\, S_0 - \cosh^2 u\, S_3^2 & \sinh u\, S_0S_1 - \mathrm{i}\cosh u\, S_3S_2 \end{pmatrix} \end{aligned} \tag{3.19}$$

where $S_\pm = S_1 \pm \mathrm{i}S_2$ and $\Delta(u)$ is a quantum determinant of the operator $\widehat{L}(u)$ (3.14). The operators $K_{BC}(u)$, $\widehat{K}_{BC}(u)$ and shifted operators $\widehat{K}_D(u - \frac{1}{2}\eta)$, $\widehat{K}_D(u - \frac{1}{2}\eta)$ obey the RE (2.10) with $R = R(u - v)$ and $S = R(u + v - \eta)$, where $R$ is the corresponding $R$-matrix (2.14). The operators $K_{BC}$ and $\widehat{K}_{BC}$ were considered in [16, 3] and the operator $K_D$ was introduced in [11].

Following [16, 3] we present some Hamiltonians in terms of the spin operators $s_j$ (3.15). They are constructed by the rule (3.4) with the matrices $K_A$, $K_{BC}$, $K_D$ and $\widehat{K}_A$, $\widehat{K}_{BC}$, $\widehat{K}_D$, and for an open chain [16, 3] read as

$$H_A = \sum_{k=1}^{N} s_1^{(k)} s_1^{(k+1)} + s_2^{(k)} s_2^{(k+1)} + s_3^{(k)} s_3^{(k+1)}$$

$$H_{BC} = H_A + \alpha s_+^{(1)} + \beta s_-^{(1)}$$

$$H_D = H_A + s_1^{(N+1)} \sum_{k=1}^{N} s_1^{(k)}$$

$$\widehat{H}_A = \sum_{k=1}^{N} s_1^{(k)} s_1^{(k+1)} + \sinh\eta\, s_3^{(k)} s_3^{(k+1)}$$

$$\widehat{H}_{BC} = \widehat{H}_A + \sinh\eta \left(\alpha s_+^{(1)} + \beta s_-^{(1)}\right)$$

$$\widehat{H}_D = \widehat{H}_A + \sinh\eta\, s_1^{(N+1)} \sum_{k=1}^{N} s_1(k).$$

In [16] operators like $K_{BC}$ were introduced for an $XYZ$ magnet chain and for the non-linear Schrödinger equation. For these systems operators similar to $K_D$ have not yet been considered.

We do not know of examples of the quadratic $R$-matrix algebras when one modifies the initial algebra and $S$ matrices depending on dynamical variables. Some interesting examples of such deformations for the linear $r$-matrix algebras will be considered in the next two sections.

## 4. Deformations of the linear $r$-matrix algebras

In this section we consider three Lie–Poisson algebras connected with different $R$-matrix algebras [4, 5, 17–19]. The Lie–Poisson brackets

$$\{\overset{1}{L}(\lambda), \overset{1}{L}(\lambda)\} = [r(\lambda,\mu), \overset{1}{L}(\lambda) + \overset{2}{L}(\mu)] \tag{4.1}$$

is a linear classical limit of the FCR (2.1) by the $R(u) = I + \mathrm{i}\eta r(u) + \mathrm{O}(\eta^2)$ the $T(u) = I + \mathrm{i}\eta L(u) + \mathrm{O}(\eta^2)$, where the parameter $\eta$ is a Planck constant $[\,,\,] \to -\mathrm{i}\eta\{\,,\,\}$. By the substitution $S(u) = I + \mathrm{i}\eta s(u) + \mathrm{O}(\eta^2)$ the bracket

$$\{\overset{1}{L}(\lambda), \overset{2}{L}(\mu)\} = [r(\lambda,\mu), \overset{1}{L}(\lambda) + \overset{2}{L}(\mu)] + [s(\lambda,\mu), \overset{1}{L}(\lambda) - \overset{2}{L}(\mu)] \tag{4.2}$$

*is related to the* RE *(2.10). The linear limit of the* GRE *(2.11) is*

$$\begin{aligned}\{\overset{1}{L}(\lambda), \overset{2}{L}(\mu)\} &= [r(\lambda,\mu), \overset{1}{L}(\lambda) + \overset{2}{L}(\lambda)] + [s(\lambda,\mu), \overset{1}{L}(\lambda) - \overset{2}{L}(\mu)] \\ &\quad + \{t(\lambda,\mu), \overset{1}{L}(\lambda) + \overset{2}{L}(\mu)\}_+ + \{w(\lambda,\mu), \overset{1}{L}(\lambda) - \overset{2}{L}(\mu)\}_+ \,.\end{aligned} \tag{4.3}$$

Here

$$\begin{gathered} A(u) = I + \mathrm{i}\eta a(i) + \mathrm{O}(\eta^2) \quad \ldots \quad D(u) = I + \mathrm{i}\eta d(u) + \mathrm{O}(\eta^2) \\ r = \frac{a+d}{2} \qquad s = \frac{b+c}{2} \qquad t = \frac{a-d}{2} \qquad w = \frac{b-c}{2} \end{gathered} \tag{4.4}$$

and $[\,,\,]$ and $\{\,,\,\}_+$ stand for a matrix commutator and anticommutator respectively. We also use the standard notations $\overset{1}{L}(\lambda) = L(\lambda) \otimes I$, $\overset{2}{L}(\mu) = I \otimes L(\mu)$ introduced for the quadratic algebras. The brackets (4.2), (4.3) define the Lie–Poisson algebras if the matrices $r(\lambda)$, $s(\lambda)$, $t(\lambda)$ and $w(\lambda)$ satisfy some modified Yang–Baxter equations [19, 18]. For simplicity we restrict ourselves to the $r$-matrix algebra (4.1) and the $rs$-matrix algebra (4.2) in the two-dimensional auxiliary space only.

Let the 'vacuum' operator $L_0$

$$L_0(\lambda) = \sum_{k=1}^{3} l_i(\lambda)\sigma_k = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}(u) \tag{4.5}$$

obeys the linear $r$-matrix algebra (4.1) with the $r$-matrix

$$r(\lambda) = \sum_{k=1}^{3} w_k(\lambda)\sigma_k \otimes \sigma_k \tag{4.6}$$

where $u_k(\lambda)$ are functions of a spectral parameter only and $\sigma_k$ are the Pauli matrices. We will require that the $r$-matrix obeys the classical Yang–Baxter equation (CYBE) and is antisymmetric, $r(\lambda) = -r(-\lambda)$ [4].

Let us introduce a deformation of the 'vacuum' $L_0(\lambda)$ operator (4.5) of the form

$$L(\lambda) = \begin{pmatrix} a & b \\ F(b,\lambda) + c & -a \end{pmatrix}(\lambda) \tag{4.7}$$

where $F(b,\lambda)$ is a not yet defined function of the matrix entry $b(\lambda)$ and the spectral parameter $\lambda$.

*Theorem 1.* The $L(\lambda)$ operator (4.7) satisfies the linear $rs$-matrix algebra (4.2), if $w_1 = w_2 \equiv w$ and the function $F(b,\lambda)$ has the form

$$F(b,\lambda) = -f(\lambda)b^{-1}(\lambda)$$

where $f(\lambda)$ is a function of the spectral parameter $\lambda$ only. The corresponding matrix $s(\lambda,\mu)$ is given by

$$s(\lambda,\mu) = \alpha(\lambda,\mu)\sigma_- \otimes \sigma_- \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where $\sigma(\lambda,\mu)$ is

$$\alpha(\lambda,\mu) = w(\lambda-\mu)\left(\frac{\partial F}{\partial b}(\lambda) - \frac{\partial F}{\partial b}(\mu)\right)$$
$$\equiv -\, w(\lambda-\mu)\left(f(\lambda)b^{-2}(\lambda) - f(\mu)b^{-2}(\mu)\right). \tag{4.8}$$

The proof is based on a direct but cumbersome computation, and is omitted.

We will use $r$-matrix $XXX$ and $XXZ$ types only [4]. These $r$-matrices are

$$r(\lambda) = \frac{\eta}{\lambda}\sum_{k=1}^{3}\sigma_k \otimes \sigma_k \qquad \text{for the } XXX \text{ case} \tag{4.9}$$

$$r(\lambda) = \frac{\eta}{\sinh\lambda}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \cosh\lambda\, \sigma_3 \otimes \sigma_3) \qquad \text{for the } XXZ \text{ case.} \tag{4.10}$$

As for the quadratic algebras, one can not use the deformation (4.7) for the linear $r$-matrix of the $XYZ$ type, because $w_1 \neq w_2$.

In this section we use the notation $d(\lambda)$ for the determinant of the $L$-operator, because the quantum determinant $\Delta(u)$ is a Casimir operator for the quadratic $R$-matrix algebras, but for the linear $r$-matrix algebras the determinant $d(\lambda)$ is a generating function of the integrals of motion. It follows from the algebra (4.2) that the function $d(\lambda) \equiv \det L(\lambda)$ can be taken as a generating function of the integrals of motion, since

$$\{d(\lambda), d(\mu)\} = 0 \qquad \lambda,\mu \in \mathbb{C}. \tag{4.11}$$

The determinant of the modified operator $L(\lambda)$ is $d(\lambda) = d_0(\lambda) - f(\lambda)$. We deform our system in such a way that new integrals of the motion differ from the old ones by some constants

$$I_{\text{new}} = I_{\text{old}} + f_k \qquad f_k \in \mathbb{C}.$$

Let the entries of the matrix $L_0(\lambda)$ be defined as the absolutely convergent Laurent series for the $XXX$ model, or the Fourier series for the $XXZ$ model of the parameter $\lambda$ or their terms

$$a(\lambda) = \sum_k a_k \lambda^k \qquad \text{for the } XXX \text{ model}$$

$$a(\lambda) = \sum_k a_k \exp(k\lambda) \qquad \text{for the } XXZ \text{ model.}$$

We can introduce a new deformation of the 'vacuum' $L_0(\lambda)$ operator (4.5), which has the form

$$L_N(\lambda) = \begin{pmatrix} a & b \\ F_N(b,\lambda) + c & -a \end{pmatrix}(\lambda) \tag{4.12}$$

where $F_N(b, \lambda)$ is a function of the spectral parameter $\lambda$ and the entry $b(\lambda)$, and reads

$$F_N(b,\lambda) = [f_N(\lambda) b^{-1}(\lambda)]_+ \tag{4.13}$$

$$f_N(\lambda) = \sum_{k=0}^{N} f_k \lambda^k \qquad \text{or} \qquad f_N(\lambda) = \sum_{k=-N}^{N} f_k \exp(k\lambda). \tag{4.14}$$

Here the brackets $[\ ]_+$ denote the standard (or Taylor) projection

$$\begin{aligned} [z]_+ &= \left[ \sum_{k=-\infty}^{+\infty} z_k \lambda^k \right]_+ \equiv \sum_{k=0}^{+\infty} z_k \lambda^k \\ [z]_+ &= \left[ \sum_{k=-\infty}^{+\infty} z_k \exp(k\lambda) \right]_+ \equiv \sum_{k=-N}^{+N} z_k \exp(k\lambda) \end{aligned} \tag{4.15}$$

for $r$-matrices of the $XXX$ and $XXZ$ types, respectively.

Note that we can also use a more general projection $[\ ]_{MN}$ (Laurent projection)

$$\begin{aligned} [z]_{MN} &= \left[ \sum_{k=-\infty}^{+\infty} z_k \lambda^k \right]_{MN} \equiv \sum_{k=-N}^{+N} z_k \exp(k\lambda) \\ [z]_{MN} &= \left[ \sum_{k=-\infty}^{+\infty} z_k \exp(k\lambda) \right]_{MN} \equiv \sum_{k=-M}^{M} z_k \exp(k\lambda). \end{aligned} \tag{4.16}$$

*Corollary 1.* The $L_N(\lambda)$ operator satisfies the linear $rs$-matrix algebra (4.2), if $w_1 = w_2 \equiv w$, and the matrix $s_N(\lambda, \mu)$ is given by

$$s(\lambda,\mu) = \alpha_N(\lambda,\mu)\sigma_- \otimes \sigma_- \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The function $\alpha_N(\lambda, \mu)$ is defined by

$$\begin{aligned} \alpha_N(\lambda,\mu) &= w(\lambda-\mu)\left[ \frac{\partial F}{\partial b}(\lambda) - \frac{\partial F}{\partial b}(\mu) \right]_+ \\ &\equiv -w(\lambda-\mu)\left( [f(\lambda)b^{-2}(\lambda)]_+ - [f(\mu)b^{-2}(\mu)]_+ \right). \end{aligned} \tag{4.17}$$

The proof is based on a straightforward calculation.

Note that now $d(\lambda) - d_0(\lambda) + bF_N \neq d_0(\lambda) - f(\lambda)$, and therefore the integrals of motion of the deformed system are functionally different from their undeformed counterparts.

Let us rewrite equations (4.1) and (4.2) in the form [1]

$$\{(\overset{1}{L}(\lambda)) \overset{\otimes}{,} (\overset{2}{L}(\mu))\} = [r_{12}(\lambda, \mu), \overset{1}{L}(\lambda)] + [r_{21}(\lambda, \mu), \overset{2}{L}(\mu)] \tag{4.18}$$

where $r_{12}(\lambda, \mu) - Pr_{12}(\mu, \lambda)P$ and the operator $P$ is a standard permutation of the auxiliary spaces [1]. The matrices $r_{12} = r_{21} = r$ stand for the $r$-matrix algebra (4.1), and $d_{1,2} = r \pm s$ stands for the $rs$-matrix algebras (4.2). We can also consider the Poisson structure (4.18) for the powers of the $L$-operator

$$\begin{gathered}\{\overset{1}{L}{}^n(\lambda) \overset{\otimes}{,} \overset{2}{L}{}^m(\mu)\} = [r_{12}^{(n,m)}(\lambda, \mu), \overset{1}{L}(\lambda)] - [r_{21}^{(n,m)}(\lambda, \mu), \overset{2}{L}(\mu)] \\ r_{ij}^{(n,m)} = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \overset{1}{L}{}^{n-k-1} \overset{2}{L}{}^{m-l-1} r_{ij} \overset{1}{L}{}^k \overset{2}{L}{}^l.\end{gathered} \tag{4.19}$$

As an immediate consequence of (4.18) and (4.19) we arrive at

$$\{\mathrm{tr}_1(\overset{1}{L}{}^n), \mathrm{tr}_2(\overset{2}{L}{}^m)\} = 0 \qquad n, m = 1, 2, \ldots . \tag{4.20}$$

and the integrals of the motion are

$$H_n(\lambda) = \mathrm{tr}_j(L_j^n) \qquad j = 1, 2, \; n = 1, 2, \ldots . \tag{4.21}$$

Note that

$$d(\lambda) \equiv \det L(\lambda) = \tfrac{1}{2}\,\mathrm{tr}\,(L^2) = \tfrac{1}{2} H_2(\lambda).$$

The Lax representation corresponding to the Hamiltonian $H_n$ (4.21) (see the work [1]) reads

$$\dot{L(\mu)} = \{H_n(\lambda), L(\mu)\} = [M_n(\mu, \lambda), L(\mu)] \tag{4.22}$$

where the matrix $M_n(\mu, \lambda)$ is determined by

$$M_n(\mu, \lambda) = n\,\mathrm{tr}_1(\overset{1}{L}{}^{n-1} r_{21}) \qquad n = 1, 2, \ldots .$$

To prove this we should take into account (4.19)

$$\dot{L} \equiv \{H_n, L\} = \mathrm{tr}_1\{\overset{1}{L}{}^n, \overset{2}{L}\} = \mathrm{tr}_1[r_{12}^{(n)}, \overset{1}{L}] + \mathrm{tr}_1[r_{21}^{(n)}, \overset{2}{L}].$$

Here the first term is zero as trace over the first space of the commutator and therefore

$$\begin{aligned} M_n(\mu, \lambda) &= \mathrm{tr}_1\, r_{21}^{(n)} = \mathrm{tr}_1 \sum_{k=0}^{n-1} (\overset{1}{L}{}^{n-k-1} r_{21} \overset{1}{L}{}^k) \\ &= \sum_{k=0}^{n-1} \mathrm{tr}_1(\overset{1}{L}{}^{n-1} r_{21}) = n\,\mathrm{tr}_1(\overset{1}{L}{}^{n-1} r_{21}) \end{aligned}$$

where cyclic permutation under the trace operation is used.

The Hamiltonians $H_n(\lambda)$ (4.21) are functions of a spectral parameter. In order to introduce the Hamiltonian $H$, which does not depend on a spectral parameter, one needs a projection

$$H = \tfrac{1}{2}\Phi_\lambda[H_2(\lambda)] = \Phi_\lambda[d(\lambda)] = \tfrac{1}{2}\Phi_\lambda[\mathrm{tr}_1 \overset{1}{L}{}^2(\lambda)] \tag{4.23}$$

where $\Phi_\lambda$ is a linear function on the spectral space, for instance

$$\Phi_\lambda[z] = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda^n z(\lambda))\bigg|_{\lambda=0}. \tag{4.24}$$

With the help of the algebra (4.2) and equations (4.18)–(4.23) we construct a Lax representation for the $L$-operators (4.5), (4.7) with the Hamiltonian (4.23)

$$\begin{aligned} \dot{L}(\mu) &= \{H, L(u)\} = \{\tfrac{1}{2}\Phi_\lambda[\mathrm{tr}_1 \overset{1}{L}{}^2(\lambda)], L(\mu)\} = [M(\mu), L(\mu)] \\ M(\mu) &= \Phi_\lambda[\mathrm{tr}_1(\overset{1}{L}(\lambda) r_{21}(\lambda,\mu))]. \end{aligned} \tag{4.25}$$

For the 'vacuum' $L_0$-operator (4.5) one obtains $r_{21} = r$ and

$$M_0 = \Phi_\lambda\left[2\sum_{k=1}^{3} l_k(\lambda) w_k(\lambda-\mu)\sigma_k\right] \tag{4.26}$$

where the natural notation $M_0$ is used for the second matrix in the Lax representation with the 'vacuum' operator $L_0$.

As an example we consider the special case of 'vacuum' $L_0$ operators and functionals $\Phi_\lambda$, which results in

$$M_0 = \sigma_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.27}$$

It is a rather strong restriction on the $L_0$ operator (4.5) and the Hamiltonian (4.23). As an immediate consequence of the Lax representation (4.25) with the matrix $M_0$ (4.27) we can rewrite the operator $L_0$ in the form

$$L_0(\mu) = \begin{pmatrix} -\frac{1}{2}b_x & b \\ -\frac{1}{2}b_{xx} & \frac{1}{2}b_x \end{pmatrix} \qquad b_x \equiv \{H_0, b\} \tag{4.28}$$

where $H_0$ is a Hamiltonian corresponding to $L_0$. The equations of motion are constructed from the equation $\{d(\lambda), H\} = 0$. They follow from the formula (4.28) and are consistent with the Lax representation (4.25)

$$\partial_x^3 b = b_{xxx} = 0. \tag{4.29}$$

For the fixed operator $L_0(\lambda)$ and projector $\Phi_\lambda$ we can consider an $L$ operator modified by the rule (4.7). In this case the matrix $M(\mu)$ in the Lax representation with the modified $L(\mu)$ operator (4.7) is constructed by the rule (4.25), where $r_{21}(\lambda,\mu) = r - s$, which gives

$$M(\mu) = \begin{pmatrix} 0 & 1 \\ -u(\mu) & 0 \end{pmatrix}. \tag{4.30}$$

Here we have imposed the property on the linear functional $\Phi_\lambda$ that defines a function $u(\mu)$

$$\Phi_\lambda\big[w_1\big(-f(\lambda)b^{-1}(\lambda)+c(\lambda)+(f(\lambda)b^{-2}(\lambda)-f(\mu)b^{-2}(\mu))b(\lambda)\big)\big]$$
$$=f(\mu)b^{-2}(\mu)=u(\mu).$$

One can apply a deformation (4.7) to the operator $L_0$ (4.28) that gives rise to an operator

$$L(\lambda)=\begin{pmatrix} -\frac{1}{2}b_x & b \\ -b(\lambda)u(\lambda)-\frac{1}{2}b_{xx} & \frac{1}{2}b_x \end{pmatrix} \qquad b_x\equiv\{H,b\} \tag{4.31}$$

where $H$ is a corresponding Hamiltonian. The equation of motion now reads

$$(\tfrac{1}{4}\partial_x^3+u\partial_x+\tfrac{1}{2}u_x)b=B_1[u]b=0 \tag{4.32}$$

where $B_1$ is the Hamiltonian operator of the first Hamiltonian structure for the coupled KdV equation.

Where $u(\lambda)$ is a rational function of the spectral paramter this equation has been investigated in many papers. Some of its solutions with $r$-matrices of the $XXX$ type are considered from the viewpoint of the $rs$–Lie–Poisson structure in the works [7, 14].

## 5. Examples of the $rs$-matrix algebra for integrable systems

A special form of the 'vacuum' $L_0$ operator (4.5) has been considered in [7, 14]. The operator $L_0$ was taken in the form (4.28), where the meromorphic function $b(\lambda)$

$$b(\lambda)=\sum_{k=1}^{n}\frac{x_k^2}{\lambda-\lambda_k}+\sum_{k=n}^{m}\lambda^{n-k}\sum_{j=n}^{m-n-k}x_jX_{n-k+j} \tag{5.1}$$

depends only on the coordinates of particles, $x_j$ being the coordinate of the $j$th particle. The $L_N$ operators defined by the rule (4.7), (4.31) are related to the restricted flows for KdV [14] and to the motion on real Riemannian spaces of constant curvature [7]. Among the dynamical models studied in [7, 14] there is a Henon–Heiles system of type (ii). Its Hamiltonian ($A$, $B$, $\varepsilon$ are constant)

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+\tfrac{1}{2}(Ax^2+By^2)+x^2y+\varepsilon y^3$$

has been extensively studied both in non-integrable and integrable regimes. The integrability holds only for the following three sets of parameters

| | | |
|---|---|---|
| (i) | $A=B$ | $\varepsilon=\frac{1}{3}$ |
| (ii) | | $\varepsilon=2$ |
| (iii) | $16A=B$ | $\varepsilon=\frac{16}{3}$. |

In this section we consider a class of integrable systems with one general property, namely, each system from this class is linearized on the Jacobi variety $\Gamma=\otimes\Gamma_j$, where $\Gamma_j$ are hyper-elliptic curves [21]. The Henon–Heiles system in cases (i) and (iii) belong to this class.

Let us write an initial system in the variables $(p_j, q_j)$, $j = 1, \ldots, n$, where $\{p_j, q_k\} = d_{jk}$. We will assume that there exists a canonical transformation

$$u_k = U_k(q_1, \ldots, q_n; p_1, \ldots, p_n) \quad v_k = V_k(q_1, \ldots, q_n; p_1, \ldots, p_n) \quad k = 1, \ldots, n \tag{5.2}$$

such that in the new variables $(v_k, u_k)$ $(\{v_j, u_k\} = d_{jk})$ equations of motion are separated and have the form

$$v_k^2 = G_k(u_k) \tag{5.3}$$

where the functions $G_k$ are given by the Laurent sets

$$G_k(u) = \sum g_j^k u^j. \tag{5.4}$$

We associated to such a system the $L$-matrix in a special form

$$L(\lambda) = \oplus_k^n L_N^{(k)}(\lambda, v_k, u_k) \tag{5.5}$$

that acts in the extended auxiliary space

$$v_{\text{aux}} = \oplus_k V_{\text{aux}}^{(k)} \tag{5.6}$$

and the matrices $L_N^{(k)}(\lambda, v_k, u_k)$ are equal to

$$L_N^{(k)}(\lambda, v_k, u_k) = \begin{pmatrix} -v_k & \lambda - u_k \\ \left[\dfrac{f_N(\lambda)}{(\lambda - u_k)}\right]_+ & v_k \end{pmatrix} = \begin{pmatrix} -v_k & \lambda - u_k \\ c_N(\lambda, u_k) & v_k \end{pmatrix}. \tag{5.7}$$

There $L_N^{(k)}(\lambda, v_k, u_k)$ matrices are the deformations of the special 'vacuum' matrices $L_0(\lambda, v_k, u_k)$

$$L_0(\lambda, v_k, u_k) = \begin{pmatrix} -v_k & \lambda - u_k \\ 0 & v_k \end{pmatrix} \tag{5.8}$$

which obey the standard linear $r$-matrix algebra (4.1) with the $r$-matrix of the $XXX$ type [4].

Thus we associate to our system $L$-matrix, which has a block structure, each block $L_N^{(k)}$ obeying an $rs$-algebra with a common $r$-matrix and different matrices $s_k$ constructed by the rule (4.17). The entries $c_N(\lambda, u_k)$ of the matrices $L_N(\lambda, u_k, v_k)$ (5.7) are polynomials of two variables $\lambda$ and $u_k$

$$c_N(\lambda, u_k) = \left[\sum_{i=0}^{N} f_i^{(k)} \lambda^i \sum_{j=0}^{+\infty} \frac{u_k^j}{\lambda^{j+1}}\right]_+ = \sum_{i=0}^{N-1} \lambda^i \sum_{j=0}^{N-1-i} u_k^i f_{j+i+1}^{(k)}. \tag{5.9}$$

Remember that the brackets $[\ ]_+$ denote a Taylor projection by the rule (4.15). The determinants $d_N^{(k)}$ of the matrices $L_N^{(k)}(\lambda, u_k, v_k)$ (5.7) are equal to

$$d_N^{(k)} = \sum_{i=1}^{N} f_j^{(k)} (u_k^j - \lambda^j) - v_k^2 \tag{5.10}$$

and the functions $G_k$ (5.3) are defined by $g_j^k = f_j^k$ (5.4).

A generating function of the integrals of motion can be taken as a determinant of the $L$-matrix (5.5) $d(\lambda) \equiv \det L(\lambda) = \prod_{k=1}^n d_N^{(k)}$. Hamiltonians for these systems can be defined by

$$H_N^n = \frac{\mathrm{d}^{(n-1)N}}{\mathrm{d}\lambda^{(n-1)N}} d(\lambda)\bigg|_{\lambda=0} \tag{5.11}$$

and their explicit form to within a constant factor reads

$$\begin{aligned} H_N^n &= \sum_{k=1}^n \left[f_N^{(k)}\right]^{-1} v_k^2 - \sum_{k=1}^n \left[f_N^{(k)}\right]^{-1} \sum_{j=1}^N f_j^{(k)} u_k^j \\ &= T^{(n)} + \beta V_N^{(n)} \qquad \beta \in \mathbb{R} \end{aligned} \tag{5.12}$$

where $T^{(k)}$ is a kinetic energy and $V_N^{(n)}$ is a potential. If higher coefficients $f_N^{(k)}$ of the polynomialos $f^{(k)}(\lambda)$ are the same for all particles $f_N^{(k)} = f_N^{(j)}$ for all $k, j$, then the Hamiltonians (5.11) can be rewritten as

$$H_N^n = \sum_{k=1}^n d_k(\lambda)\bigg|_{\lambda=0} . \tag{5.13}$$

For the Laurent projection $[\ ]_{MN}$ (4.16) the Hamiltonians (5.12) are equal to

$$H_N^n = \sum_{k=1}^n \left[f_N^{(k)}\right]^{-1} \left( v_k^2 - \sum_{j=-(M-1)}^{N} f_j^{(k)} u_k^j \right) . \tag{5.14}$$

The operator $L_0$ (5.8) and the modified operator $L_N$ (5.7) have a hidden internal structure [10]

$$\begin{aligned} L_0(\lambda, v_k, u_k) &= \begin{pmatrix} -v_k & \lambda - u_k \\ 0 & v_k \end{pmatrix} = \begin{pmatrix} -\sum \alpha_j p_j^{(k)} & \lambda - \sum \alpha_j^{-1} q_j^{(k)} \\ 0 & \sum \alpha_j p_j^{(k)} \end{pmatrix} \\ L_N(\lambda, v_k, u_k) &= \begin{pmatrix} -\sum \alpha_j p_j^{(k)} & \lambda - \sum \alpha_j^{-1} q_j^{(k)} \\ \left[\dfrac{F_N(\lambda)}{\lambda - \sum \alpha_j^{-1} q_j^{(k)}}\right]_+ & \sum \alpha_j p_j^{(k)} \end{pmatrix} \end{aligned} \tag{5.15}$$

where the variables $v_k$ and $u_k$ can be considered as linear combinations of some canonical variables $p_j^{(k)}$, $q_j^{(k)}$ $j = 1, \ldots, K$. If we consider $n$-particle systems only and require that the Hamiltonian (5.12) has a canonical form with a kinetic energy

$$T^{(n)} = \sum_{k=1}^n v_k^2 = \sum q_j^2$$

then all the internal structure (5.15) is reduced to the Jacobi transformations for the $n$ particles.

We illustrate this scheme by the simplest cases of the two- and three-particle systems under the Jacobi transformations. For two-particle systems, after the transformation $u_1 = q_1 + q_2$, $u_2 = q_1 + q_2$, the uniform potentials $V_N^{(2)}$ of the degrees $j = 1, 2, \ldots, N$ read

$$
\begin{aligned}
V_1^{(2)} &= e_1^+ q_1 + e_2^+ q_2 \\
V_2^{(2)} &= V_1^{(2)} + e_2^+(q_1^2 + q_2^2) + e_2^- q_1 q_2 \\
V_3^{(2)} &= V_2^{(2)} + e_3^+(q_1^3 + 3q_1 q_2^2) + e_3^-(q_2^3 + 3q_1^2 q_2) \\
V_N^{(2)} &= \sum_{j=1}^{N} \sum_{i=0}^{j} C_j^i \left( f_j^{(1)} + (-1)^i f_j^{(2)} \right) q_1^{j-1} q_2^i
\end{aligned}
\tag{5.16}
$$

where $e_N^{\pm} = (f_N^{(1)} \pm f_N^{(2)})$ and $C_j^i$ are the binomial coefficients.

For the three-particle systems with equal masses we can choose the coefficients $f_N^{(k)}$ in such a way that the Jacobi transformations (5.2) have the simplest form

$$
u_1 = (q_1 - 2q_2 + q_3) \qquad u_2 = -3(q_1 - q_3) \qquad u_3 = (q_1 + q_2 + q_3)
$$

with $f_N^{(1)} = 6$, $f_N^{(2)} = 18$, $f_N^{(3)} = 3$. The first uniform potentials $V_N^{(3)}$ of the degree $j = 1, 2, \ldots, N$ are

$$
\begin{aligned}
V_1^{(3)} &= (f_1^{(1)} - 3f_1^{(2)} + f_1^{(3)})q_1 + (f_1^{(3)} - 2f_1^{(1)})q_2 + (f_1^{(1)} + 3f_1^{(2)} + f_1^{(3)})q_3 \\
V_2^{(3)} &= V_1^{(2)} + (f_2^{(1)} + 9f_2^{(2)} + f_2^{(3)})(q_1^2 + q_3^2) + (4f_2^{(1)} + f_2^{(3)})q_2^2 \\
&\quad + (2f_2^{(3)} - 4f_2^{(1)})(q_1 q_2 + q_2 q_3) + 2(f_2^{(1)} - 9f_2^{(2)} + f_2^{(3)})q_1 q_3 \\
V_N^{(3)} &= V_{N-1}^{(3)} + \sum_{j+k+l=N} f_{jkl} q_1^j q_2^k q_3^l
\end{aligned}
\tag{5.17}
$$

where the coefficients $f_{jkl}$ are expressed through $f_j^{(k)}$ and the binomial coefficients $C_j^i$.

For the two-particle systems the Hamiltonian $H_3^{(2)}$ (5.12), (5.16) coincides with the Hamiltonian of the Henon–Heiles system of type (i) and the corresponding $L$ operator (5.5) has been considered in [20].

The Henon–Heiles system of type (iii) can be embedded in the scheme developed after a more complicated canonical transformation.

*Proposition 1.* The change of the canonical variables $v_k$, $u_k$, $k = 1, 2$, into the variables $x$, $p_x$ and $y$, $p_y$ under the rule

$$
\begin{aligned}
x^2 &= \alpha \left. \frac{d_1(\lambda) - d_2(\lambda)}{b_1(\lambda) - b_2(\lambda)} \right|_{\lambda=0} \qquad \alpha \in \mathbb{R} \\
p_x &= \left. \frac{a_1(\lambda) - a_2(\lambda)}{b_1(\lambda) - b_2(\lambda)} \right|_{\lambda=0} x \\
y &= \beta \left. (b_1(\lambda) + b_2(\lambda)) \right|_{\lambda=0} - \tfrac{1}{2}\alpha\beta \left( \frac{p_x}{x} \right)^2 \qquad \beta \in \mathbb{R} \\
p_y &= \frac{1}{\beta} \left. (a_1(\lambda) + a_2(\lambda)) \right|_{\lambda=0} + \alpha\beta \frac{p_x}{x} \left( 1 + \frac{p_x^2}{x^2} \right)
\end{aligned}
\tag{5.18}
$$

where $a_k(\lambda)$ and $b_k(\lambda)$ are the entries of the matrices $L_N(\lambda, v_k, u_k)$ $k = 1, 2$ (5.7), is a canonical transformation.

*Proof.* We fix a variable $x^2$ and the Hamiltonian $H = (d_1 + d_2)(\lambda = 0)$ by (5.13). Then a corresponding momentum $p_x = \{H, x\}$ follows from the $rs$-algebra (4.2)

$$\begin{aligned}
2xp_x &= \{H, x^2\} \\
&= \alpha \left\{ d_1(\lambda) + d_2(\lambda), \frac{d_1(\mu) - d_2(\mu)}{b_1(\mu) - b_2(\mu)} \right\}\Bigg|_{\mu=0,\lambda=0} \\
&= -\alpha \frac{d_1(\mu) - d_2(\mu)}{(b_1(\mu) - b_2(\mu))^2} \{d_1(\lambda) + d_2(\lambda), b_1(\mu) - b_2(\mu)\}\Bigg|_{\mu=0,\lambda=0} \\
&= \alpha \frac{d_1(\mu) - d_2(\mu)}{b_1(\mu) - b_2(\mu)} \frac{a_1(\mu) - a_2(\mu)}{b_1(\mu) - b_2(\mu)}\Bigg|_{\mu=0} \\
&= \frac{a_1(\mu) - a_2(\mu)}{b_1(\mu) - b_2(\mu)}\Bigg| + \mu = 0x
\end{aligned}$$

where we have used the explicit form of the $L_N$ operators (5.7) and the technique developed by Sklyanin [17]. The variable $y$ is fixed by the condition $\{x, y\} = 0$ and a corresponding momentum $p_y = \{H, y\}$ is calculated from the $rs$-algebra and the Hamilton–Jacobi equations.

Here we have used the following relations

$$\{b(\lambda), a(\mu)\} = \frac{1}{\mu - \lambda} (b(\mu) - b(\lambda))$$

$$\{b(\lambda), (\mu)\} = \frac{2}{\mu - \lambda} (a(\mu) - a(\lambda)) = 0$$

$$\{b(\lambda), b(\mu)\} = 0$$

which are determined by the $r$-matrix only and a relation $\{d_k(\lambda), d_j(\mu)\} = 0$, which is given the $rs$-algebra (4.2) with $s_k$-matrices (4.17).

To describe the Henon–Heiles system of type (iii) let us apply this transformation to the $L_3(\lambda, v_k, u_k)$ matrices (5.7) with the same third power non-linearity, as was done in the case (i). After this transformation we obtained the $l$ operator (5.5), which has been investigated in [20] for the special choice of the function $f_3(\lambda)$

$$f_3(\lambda) = -\tfrac{1}{6}(\lambda^3 - \tfrac{1}{2}A\lambda^2 - \tfrac{3}{2}A^2\lambda). \tag{5.19}$$

We can not prove that after the transformation (5.18) we will arrive at the Hamiltonians (5.12) in the natural form $H = T + V$, without some additional assumptions, for instance some constraints on the functions $f_N^{(k)}(\lambda)$.

For the two-particle system the transformation (5.2) is a special case of a wide class of canonical transformation. For instance, we can use the canonical transformation of the variables $u$, $p_u$ and $v$, $p_v$ to the variables $x$, $p_x$ and $y$, $p_y$ by the rule

$$
\begin{aligned}
x^\alpha &= \sum_{j=0}^{P} w_j u^j + \gamma v^\beta - \frac{p_u p_v}{v^{\beta-1}} \qquad \alpha, \beta, \gamma, w_j \in \mathbb{R} \\
p_x &= \frac{\alpha}{(\beta-1)\gamma} \frac{p_v}{v^{\beta-1}} x^{\alpha-1} \\
y &= 2(\beta-1)\gamma u - \left(\frac{p_v}{v^{\beta-1}}\right)^2 \\
p_y &= \frac{1}{2(\beta-1)\gamma} \left( p_u - \frac{1}{\alpha} \sum_{j=0}^{P} j w_j \sum_{i=0}^{j/2} C_j^{2i} y^{j-2i} (2i+1)^{-1} \left(\frac{p_v}{v^{\beta-1}}\right)^{2i+1} \right).
\end{aligned}
\tag{5.20}
$$

This transformation extends the transformation (5.18) to arbitrary real constants $\alpha$ and $\beta$.

The complete classification of the two- and three-particle systems described in this section and their identification with the known systems will be studied elsewhere. For the two-particle systems we can consider potentials with higher powers of nonlinearity $N$ for the Taylor (4.15) and Laurent (4.16) projections by the transformations (5.18) and (5.20).

For the three-particle systems we can generalize the operator $L$ (5.5) and consider another ansatz for it, which has the block in the form (5.1) [7]

$$
L(\lambda) = L_{12}(\lambda) \oplus L_3(\lambda) \tag{5.21}
$$

where

$$
L_{12} = \begin{pmatrix} -v_1 - \dfrac{v_2 u_2}{\lambda} & \lambda - u_1 - \dfrac{u_2^2}{\lambda} \\ c_N(\lambda) - \dfrac{v_2^2}{\lambda} & v_1 - \dfrac{v_2 u_2}{\lambda} \end{pmatrix} \qquad L_3 = \begin{pmatrix} -v_3 & \lambda - u_3 \\ c_N(\lambda) & v_3 \end{pmatrix}.
$$

We can also consider the extension of the canonical transformations (5.18) and (5.20) to this case.

## 6. Conclusions

The next problem is to consider the general form of the linear $r$-matrix algebra (4.4) with four matrices $r, s, t, w$ and the deformations of the 'vacuum' operators $L_0$ (4.7), where $F(\lambda)$ is a function of a spectral parameter $\lambda$ and coordinates $x_k$, but is not a function on entry $b(\lambda)$. It will be interesting also to examine the $L$-operator for Calogero systems

$$
L_n = \frac{1}{\lambda} \begin{pmatrix} \sum_{k=1}^n p_k x_k & \sum_{k=1}^n x_k^2 \\ \displaystyle\sum_{k=1}^n p_k^2 + \sum_{k \neq j}^n \frac{1}{(x_k - x_j)^2} & -\sum_{k=1}^n p_k x_k \end{pmatrix}
$$

which satisfy the linear $r$-matrix algebra (4.1) with $r$-matrix of $XXX$ type (4.9).

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